Product Varieties in a Quality-Differentiated Goods Monopoly

by

Changying Li and Jianhu Zhang

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Online Appendix

A.1 Proof of Lemma 1

Taking the FOCs of equation (5) with respect to p_i yields the matrix $\Lambda\mathrm{p} = \mathrm{y}$, where $\mathrm{p} = (p_1, ..., p_n)^T$,

$$\Lambda = \begin{bmatrix} q_2 & -q_1 & 0 & 0 & 0 & 0 & 0 \\ q_3 - q_2 & q_1 - q_3 & q_2 - q_1 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & q_n - q_{n-1} & q_{n-2} - q_n & q_{n-1} - q_{n-2} \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}_{n\times n},$$

and

$$\mathrm{y} = \frac{1}{2}\begin{bmatrix} q_2 c(q_1) - q_1 c(q_2) \\ c(q_1)(q_3 - q_2) + c(q_2)(q_1 - q_3) + c(q_3)(q_2 - q_1) \\ \vdots \\ c(q_{n-2})(q_n - q_{n-1}) + c(q_{n-1})(q_{n-2} - q_n) + c(q_n)(q_{n-1} - q_{n-2}) \\ c(q_n) - c(q_{n-1}) + \bar{\theta}q_n - \bar{\theta}q_{n-1} \end{bmatrix}_{n\times 1}.$$

Thus we have $p_i = [c(q_i) + q_i\bar{\theta}]/2$, $i = 1, ..., n$.

It is easy to show that the Hessian matrix is negative definite, and therefore the second-order conditions (SOCs) hold. *Q.E.D.*

A.2 Proof of Lemma 2

Substituting equation (6) into equation (5) yields the profit function

$$\text{(A1)} \qquad \Pi = \frac{m}{16\bar{\theta}}\left\{4\bar{\theta}q_n(\bar{\theta} - \alpha q_n) + \alpha^2 q_n^3 + \alpha^2 \sum_{i=1}^{n-1} q_i q_{i+1}(q_{i+1} - q_i)\right\} - f(n).$$

Taking the FOCs of equation (A1) with respect to q_i yields

$$\text{(A2)} \qquad \frac{\partial \Pi}{\partial q_i} = \frac{m\alpha^2(q_{i+1} - q_{i-1})(q_{i-1} - 2q_i + q_{i+1})}{16\bar{\theta}} = 0, \quad i = 1, ..., n-1;$$

$$\text{(A3)} \qquad \frac{\partial \Pi}{\partial q_n} = \frac{m[2\bar{\theta} + \alpha(q_{n-1} - 3q_n)][2\bar{\theta} - \alpha(q_{n-1}) + q_n)]}{16\bar{\theta}} = 0.$$

Note that since $q_i < q_{i+1}$, equation (A2) implies that $q_i = iq_1$, $i = 1, ..., n$. Substituting $q_i = iq_1$ into equation (A3) leads to

$$[(2n-1)q_1\alpha - 2\bar{\theta}] \cdot [(2n+1)q_1\alpha - 2\bar{\theta}] = 0,$$

the solutions for which are

$$q_1 = \frac{2\bar{\theta}}{\alpha(2n+1)} \quad \text{and} \quad q_1 = \frac{2\bar{\theta}}{\alpha(2n-1)}.$$

It is easy to check that the Hessian matrix is negative definite only if $q_1 = 2\bar{\theta}/[\alpha(2n+1)]$. Thus, the optimal quality–price combinations are given by

$$(q_i, p_i) = \left(\frac{2i\bar{\theta}}{\alpha(2n+1)}, \frac{i(1+i+2n)\bar{\theta}^2}{\alpha(1+2n)^2}\right), \quad i = 1, ..., n.$$

The monopoly profit can be written as

$$\Pi = \frac{n(1+n)\Gamma}{3(1+2n)^2} - f(n).$$

Taking the FOC yields

$$\text{(A4)} \qquad \frac{d\Pi}{dn} \equiv G(n) = \frac{\Gamma}{3(1+2n)^3} - f^{'}(n) = 0.$$

The SOC is satisfied. Observe $\Gamma > 81 \cdot f^{'}(1)$ implies that $G(1) > 0$. Since $\lim_{n\to\infty} G(n) = -f^{'}(\infty) < 0$, there is a unique $n^* > 1$ to satisfy equation (A4). Therefore, the monopoly bundle of strategy is given by

$$\left(q_i^*, p_i^*, n^*\right) \equiv \left(\frac{2i\bar{\theta}}{\alpha(2n^*+1)}, \frac{i(1+i+2n^*)\bar{\theta}^2}{\alpha(1+2n^*)^2}, n^*\right), \quad i = 1, ..., n^*.$$

Monopoly profit, consumer surplus, and social welfare are, respectively,

$$\Pi^* = \frac{n^*(1+n^*)\Gamma}{3(1+2n^*)^2} - f(n^*), \quad CS^* = \frac{n^*(1+n^*)\Gamma}{6(1+2n^*)^2}, \quad \text{and} \quad W^* = \frac{n^*(1+n^*)\Gamma}{2(1+2n^*)^2} - f(n^*).$$

Q.E.D.

A.3 Proof of Proposition 1

The proof follows from the following three steps:

Step 1: The Derivation of the First-Best Price. The social welfare function is

$$W = \frac{m}{\bar{\theta}} \sum_{k=1}^{n} \int_{\theta_k}^{\theta_{k+1}} [\theta q_i - c(q_i)] d\theta - f(n), \quad \text{where } \theta_{n+1} = 1. \tag{A5}$$

The FOCs with respect to p_i lead to the matrix $\Lambda^F \mathrm{p}^F = \mathrm{y}^F$, where $\Lambda^F = \Lambda$ (in section A.1), $\mathrm{p}^F = (p_1^F, ..., p_n^F)^T$, and

$$\mathrm{y}^F = \begin{bmatrix} q_2 c(q_1) - q_1 c(q_2) \\ c(q_1)(q_3 - q_2) + c(q_2)(q_1 - q_3) + c(q_3)(q_2 - q_1) \\ \vdots \\ c(q_{n-2})(q_n - q_{n-1}) + c(q_{n-1})(q_{n-2} - q_n) + c(q_n)(q_{n-1} - q_{n-2}) \\ c(q_n) - c(q_{n-1}) \end{bmatrix}_{n \times 1}.$$

Thus the first-best price is

$$p_i^F = c(q_i), \quad i = 1, ..., n. \tag{A6}$$

It is easy to show that the Hessian matrix is negative definite. Hence, the SOCs hold.

Step 2: The Determinants of n^*, n^S, and n^F. Substituting equation (6) into equation (5) yields the profit function

$$\begin{aligned} \Pi &= \sum_{i=1}^{n} \frac{1}{2} \big[q_i \bar{\theta} - c(q_i)\big] x_i - f(n) \\ &= \frac{m}{4\bar{\theta}} \Bigg\{ \sum_{i=1}^{n-1} \big[q_i \bar{\theta} - c(q_i)\big] \cdot \left[\frac{c(q_{i+1}) - c(q_i)}{q_{i+1} - q_i} - \frac{c(q_i) - c(q_{i-1})}{q_i - q_{i-1}} \right] \\ &\qquad + \big[q_n \bar{\theta} - c(q_n)\big] \cdot \left[\bar{\theta} - \frac{c(q_n) - c(q_{n-1})}{q_n - q_{n-1}} \right] \Bigg\} - f(n) \\ &= \Omega(q_1, ..., q_n, n) - f(n), \end{aligned} \tag{A7}$$

where

$$\Omega(q_1, ..., q_n, n) = \frac{m}{4\bar{\theta}} \left\{ \bar{\theta}^2 q_n - 2\bar{\theta} c(q_n) + \sum_{i=0}^{n-1} \frac{[c(q_{i+1}) - c(q_i)]^2}{q_{i+1} - q_i} \right\}.$$

Substituting equation (A6) into equation (A5) yields the first-best welfare function

$$\begin{aligned} W^F = \frac{m}{\bar{\theta}} \Bigg(&\sum_{i=1}^{n-1} \int_{\frac{c(q_i) - c(q_{i-1})}{q_i - q_{i-1}}}^{\frac{c(q_{i+1}) - c(q_i)}{q_{i+1} - q_i}} [\theta q_i - c(q_i)] d\theta \\ &+ \int_{\frac{c(q_n) - c(q_{n-1})}{q_n - q_{n-1}}}^{\bar{\theta}} [\theta q_n - c(q_n)] d\theta \Bigg) - f(n) \end{aligned}$$

$$= 2\Omega(q_1, ..., q_n, n) - f(n). \tag{A8}$$

Using equation (2) and substituting equation (6) into equation (A5), we obtain the second-best welfare function,

$$W^S = \frac{m}{\bar{\theta}} \left(\sum_{i=1}^{n-1} \int_{\frac{1}{2}\left[\frac{c(q_i)-c(q_{i-1})}{q_i - q_{i-1}} + \bar{\theta}\right]}^{\frac{1}{2}\left[\frac{c(q_{i+1})-c(q_i)}{q_{i+1} - q_i} + \bar{\theta}\right]} [q_i\theta - c(q_i)]d\theta \right.$$
$$\left. + \int_{\frac{1}{2}\left[\frac{c(q_n)-c(q_{n-1})}{q_n - q_{n-1}} + \bar{\theta}\right]}^{\bar{\theta}} [q_n\theta - c(q_n)]d\theta \right) - f(n)$$
$$= 3\Omega(q_1, ..., q_n, n)/2 - f(n). \tag{A9}$$

Observing equations (A7)–(A9), we see that, given the number of varieties, n, the optimal qualities for equations (A7)–(A9) are exactly the same. Substituting the optimal qualities $(q_1(n), ..., q_n(n))$ into equations (A7)–(A9) leads to

$$\Pi(n) = \Omega(q_1(n), ..., q_n(n), n) - f(n),$$
$$W^F(n) = 2\Omega(q_1(n), ..., q_n(n), n) - f(n),$$

and

$$W^S(n) = 3\Omega(q_1(n), ..., q_n(n), n)/2 - f(n).$$

Define $\omega(n) = \Omega(q_1(n), ..., q_n(n), n)$; then the number n^* of optimal varieties under a monopoly is determined by the following equations:

$$\sum_{i=1}^{n} \frac{\partial\omega(n^*)}{\partial q_i}\frac{\partial q_i(n^*)}{\partial n} + \frac{\partial\omega(n^*)}{\partial n} - f'(n^*) = 0.$$

Using the envelope theorem, we have

$$\sum_{i=1}^{n} \frac{\partial\omega(n^*)}{\partial q_i}\frac{\partial q_i(n^*)}{\partial n} = 0$$

and

$$\frac{\partial\omega(n^*)}{\partial n} - f'(n^*) = 0. \tag{A10}$$

Similarly, the first-best and second-best varieties are respectively determined by

$$2\frac{\partial\omega(n^F)}{\partial n} - f'(n^F) = 0, \tag{A11}$$

$$\frac{3}{2}\frac{\partial\omega(n^S)}{\partial n} - f'(n^S) = 0. \tag{A12}$$

Step 3: Comparison between n^*, n^S, and n^F. Combining equations (A11) and (A12) yields

$$\left(\frac{3}{2}\frac{\partial\omega(n^S)}{\partial n} - f'(n^S)\right) - \left(\frac{3}{2}\frac{\partial\omega(n^F)}{\partial n} - f'(n^F)\right) = \frac{1}{2}\frac{\partial\omega(n^F)}{\partial n}.$$

Using the mean-value theorem and equation (A11), we have

$$\left(\frac{3}{2}\frac{d}{dn}\left(\frac{\partial\omega(n_1)}{\partial n}\right) - f''(n_1)\right)\cdot(n^S - n^F) = \frac{f'(n^F)}{4} > 0, \quad n_1 \in (n^S, n^F).$$

The SOC of equation (A12) requires that

$$\frac{3}{2}\frac{d}{dn}\left(\frac{\partial\omega(n_1)}{\partial n}\right) - f''(n_1) < 0,$$

and hence

(A13) $$n^S - n^F < 0.$$

Similarly, combining equations (A10) and (A12) yields

$$\left(\frac{\partial\omega(n^*)}{\partial n} - f'(n^*)\right) - \left(\frac{\partial\omega(n^S)}{\partial n} - f'(n^S)\right) = \frac{1}{2}\frac{\partial\omega(n^S)}{\partial n}.$$

Using the mean-value theorem and equation (A12), we have

$$\left(\frac{d}{dn}\left(\frac{\partial\omega(n_2)}{\partial n}\right) - f''(n_2)\right)\cdot(n^* - n^S) = \frac{f'(n^S)}{3} > 0, \quad n_2 \in (n^*, n^S).$$

The SOC of equation (A10) requires that

$$\frac{d}{dn}\left(\frac{\partial\omega(n_2)}{\partial n}\right) - f''(n_2) < 0,$$

and hence

(A14) $$n^* - n^S < 0.$$

Combining equations (A13) and (A14), we have $n^* < n^S < n^F$. *Q.E.D.*

A.4 Proof of Proposition 2

The proof follows from the following two steps.

A.4.1 Step 1: The Derivation of the First- and Second-Best Outcomes

(1) The Derivation of the First-Best Outcome. Given the cost function $c(q_i) = \alpha q_i^2/2$, the first-best prices are $p_i^F = c(q_i) = \alpha q_i^2/2$, $i = 1, ..., n$. Substituting these prices into equation (A5) yields the welfare function

$$W^F = \frac{m}{8\bar{\theta}}\left\{4\bar{\theta}q_n(\bar{\theta} - \alpha q_n) + \alpha^2 q_n^3 + \alpha^2 \sum_{i=1}^{n-1} q_i q_{i+1}(q_{i+1} - q_i)\right\} - f(n).$$

Applying similar calculations to those in section A.2 leads to $q_i^F = 2i\bar{\theta}/[\alpha(2n+1)]$, $i = 1, ..., n$. Substituting $q_i^F = 2i\bar{\theta}/[\alpha(2n+1)]$ into the welfare function W^F gives us

$$W^F = \frac{2n(1+n)\Gamma}{3(1+2n)^2} - f(n).$$

The FOC is

$$\text{(A15)} \qquad \frac{dW^F}{dn} = \frac{2\Gamma}{3(1+2n)^3} - f'(n) = 0.$$

Repeating a similar analysis to that in section A.2, we can show that there is a unique $n^F > 1$ to satisfy equation (A15). Therefore, the first-best outcome is given by

$$\left(q_i^F, p_i^F, n^F\right) \equiv \left(\frac{2i\bar{\theta}}{\alpha(2n^F+1)}, \frac{2i^2\bar{\theta}^2}{\alpha(1+2n^F)^2}, n^F\right), \quad i = 1, ..., n^F.$$

(2) The Derivation of the Second-Best Outcome. Section A.3 implies that, given the number of varieties, the second-best qualities and prices coincide with the monopoly solutions. Lemma 2 shows that

$$p_i^S = \frac{i(1+i+2n)\bar{\theta}^2}{\alpha(1+2n)^2} \quad \text{and} \quad q_i^S = \frac{2i\bar{\theta}}{\alpha(2n+1)}.$$

Substituting them into equation (A9) leads to

$$W^S = \frac{n(1+n)\Gamma}{2(1+2n)^2} - f(n).$$

The FOC is

$$\text{(A16)} \qquad \frac{dW^S}{dn} = \frac{\Gamma}{2(1+2n)^3} - f'(n) = 0.$$

Repeating a similar approach to that in section A.2, we can show that there is a unique $n^S > 1$ to satisfy equation (A16). Therefore, the second-best outcome is given by

$$\left(q_i^S, p_i^S, n^S\right) \equiv \left(\frac{2i\bar{\theta}}{\alpha(2n^S+1)}, \frac{i(1+i+2n^S)\bar{\theta}^2}{\alpha(1+2n^S)^2}, n^S\right), \quad i = 1, ..., n^S.$$

A.4.2 Step 2: Comparison between Quality Ranges

Lemma 2 and step 1 imply that

$$q_{n^M}^M - q_1^M = \frac{2(n^M-1)\bar{\theta}}{\alpha(2n^M+1)}, \quad M \in \{*, S, F\}.$$

Note that

$$\frac{d(q_{n^M}^M - q_1^M)}{dn^M} = \frac{6\bar{\theta}}{\alpha(1+2n^M)^2} > 0 \qquad \text{and} \qquad n^* < n^S < n^F.$$

The proof of Proposition 2 is completed, i.e., $(q_{n^*}^* - q_1^*) < (q_{n^S}^S - q_1^S) < (q_{n^F}^F - q_1^F)$.
Q.E.D.

A.5 Proof of Corollary 1

Lemma 2 and the first step of section A.4 imply that

$$q_1^M = \frac{2\bar{\theta}}{\alpha(2n^M+1)} \quad \text{and} \quad q_{n^M}^M = \frac{2n^M\bar{\theta}}{\alpha(2n^M+1)}, \qquad M \in \{*, S, F\}.$$

Note that

$$\frac{dq_1^M}{dn^M} = \frac{-4\bar{\theta}}{\alpha(2n^M+1)^2} < 0 \qquad \text{and} \qquad \frac{dq_{n^M}^M}{dn^M} = \frac{2\bar{\theta}}{\alpha(2n^M+1)^2} > 0.$$

Observe, $n^* < n^S < n^F$ implies that $q_1^* > q_1^S > q_1^F$ and $q_{n^*}^* < q_{n^S}^S < q_{n^F}^F$. *Q.E.D.*

A.6 Proof of Proposition 3

(1) Proof of Proposition 3(1a) and (1b). Let $\tau > 0$ denote the subsidy rate under variable-cost subsidy. The profit function becomes

$$\Pi(\tau) = \sum_{i=1}^{n}[p_i - (1-\tau)c(q_i)]x_i - f(n).$$

Then the optimal price is $p_i(\tau) = [(1-\tau)c(q_i) + q_i\bar{\theta}]/2$. The monopoly profit is

$$\Pi(\tau) = \frac{m}{4\bar{\theta}}\left\{\bar{\theta}^2 q_n - 2\bar{\theta}(1-\tau)c(q_n) + (1-\tau)^2\sum_{i=0}^{n-1}\frac{[c(q_{i+1})-c(q_i)]^2}{q_{i+1}-q_i}\right\} - f(n).$$

Using a similar approach to that in section A.2, we have

$$q_i(\tau) = \frac{2i\bar{\theta}}{\alpha(2n+1)(1-\tau)} \quad \text{and} \quad \Pi(\tau) = \frac{n(1+n)\Gamma}{3(1-\tau)(1+2n)^2} - f(n).$$

The number of optimal varieties, $n(\tau)$, under a monopoly is determined by

(A17) $$\frac{d\Pi(\tau)}{dn} = \frac{\Gamma}{3(1-\tau)(1+2n(\tau))^3} - f'(n(\tau)) = 0.$$

The implicit-function theorem implies that

(A18) $$\frac{dn(\tau)}{d\tau} = \frac{(1+2n(\tau))\Gamma}{3(1-\tau)[2\Gamma + (1+2n(\tau))^4(1-\tau)f''(n(\tau))]} > 0.$$

The consumer surplus and social welfare are, respectively,

(A19) $$CS(\tau) = \frac{n(\tau)[1+n(\tau)]\Gamma}{6[1+2n(\tau)]^2(1-\tau)}, \quad \text{and} \quad W(\tau) = \frac{n(\tau)[1+n(\tau)](3-5\tau)\Gamma}{6[1+2n(\tau)]^2(1-\tau)^2} - f(n(\tau)).$$

Comparing equation (A17) with (A15) yields $\tau_1 = 1/2$, i.e., $n(1/2) = n^F$. And comparing equation (A17) with (A16) yields $\tau_2 = 1/3$, i.e., $n(1/3) = n^S$.

Section A.2 shows that $n^* = n(0)$, $CS^* = CS(0)$, and $W^* = W(0)$. Observe,

$$\frac{dCS(\tau)}{d\tau} = \frac{\Gamma\big(n + 3n^2 + 2n^3 + (1-\tau)\cdot n'(\tau)\big)}{6(1-\tau)^2(1+2n)^3} > 0$$

implies that $CS(1/2) > CS(1/3) > CS(0)$. Therefore a variable-cost subsidy leads to a higher level of consumer surplus.

In the following, we compare social welfare under different cases:

$$\begin{aligned}
W(0) - W\left(\frac{1}{2}\right) &= \frac{n^*(1+n^*)\Gamma}{2(1+2n^*)^2} - f(n^*) - \frac{n^F(1+n^F)\Gamma}{3(1+2n^F)^2} + f(n^F) \\
&= \frac{n^*(1+n^*)\Gamma}{6(1+2n^*)^2} + \left(\frac{n^*(1+n^*)\Gamma}{3(1+2n^*)^2} - \frac{n^F(1+n^F)\Gamma}{3(1+2n^F)^2}\right) + \big(f(n^F) - f(n^*)\big) \\
&= \frac{n^*(1+n^*)\Gamma}{6(1+2n^*)^2} + \frac{\Gamma\cdot(n^* - n^F)}{3(1+2n_3)^3} + f'(n_4)(n^F - n^*), \\
&\quad \text{where } n_3, n_4 \in (n^*, n^F) \text{ (by the mean-value theorem)} \\
&> \frac{n^*(1+n^*)\Gamma}{6(1+2n^*)^2} - \frac{\Gamma\cdot(n^F - n^*)}{3(1+2n_3)^3} + f'(n^*)(n^F - n^*) \\
&\quad \text{(by } n_4 > n^* \text{ and } f''(\cdot) > 0) \\
&= \frac{n^*(1+n^*)\Gamma}{6(1+2n^*)^2} + \left(\frac{\Gamma}{3(1+2n^*)^3} - \frac{\Gamma}{3(1+2n_3)^3}\right)(n^F - n^*) \\
&\quad \text{(by equation (A4))} \\
&> \frac{n^*(1+n^*)\Gamma}{6(1+2n^*)^2} > 0 \\
&\quad \text{(by } n_3 > n^* \text{ and } n^F > n^*).
\end{aligned}$$

$$\begin{aligned}
W\left(\frac{1}{3}\right) - W(0) &= \left(\frac{n^S(1+n^S)\Gamma}{2(1+2n^S)^2} - f(n^S)\right) - \left(\frac{n^*(1+n^*)\Gamma}{2(1+2n^*)^2} - f(n^*)\right) \\
&= \left(\frac{\Gamma}{2(1+2n_5)^3} - f'(n_5)\right)(n^S - n^*), \\
&\quad \text{where } n_5 \in (n^*, n^S) \text{ (by the mean-value theorem)} \\
&> \left(\frac{\Gamma}{2(1+2n^S)^3} - f'(n^S)\right)(n^S - n^*) \\
&\quad \text{(by } f''(\cdot) > 0 \text{ and } n_5 < n^S) \\
&= 0 \\
&\quad \text{(by equation (A16))}.
\end{aligned}$$

(2) Proof of Proposition 3(1)(c). Let τ^W denote the subsidy rate that maximizes welfare. τ^W should satisfy $dW(\tau)/d\tau = 0$. Note that $n(1/3) = n^S$ and $n(0) = n^*$. Equations (A18) and (A19) imply that

$$\left.\frac{dW(\tau)}{d\tau}\right|_{\tau=1/3} = -\left.\frac{3n(1+n)\Gamma}{8(1+2n)^2}\right|_{n=n(1/3)} < 0$$

and

$$\left.\frac{dW(\tau)}{d\tau}\right|_{\tau=0} = \left.\frac{\Gamma\big((1+6n+6n^2)\Gamma + 3n(1+n)(1+2n)^4 f''(n)\big)}{18(1+2n)^2(2\Gamma + (1+2n)^4 f''(n))}\right|_{n=n(0)} > 0.$$

The SOC requires that $d^2W(\tau)/d\tau^2 < 0$, which implies that $0 < \tau^W < \tau_2 = 1/3 < \tau_1 = 1/2$. Note that $\tau_1 = 1/2$ is the subsidy rate that induces first-best varieties and $n(\tau)$ is an increasing function; thus, we complete the proof of Proposition 3(1)(c).

(3) Proof of Proposition 3(2), (a) and (b). Let $\sigma > 0$ denote the amount of subsidy per unit of fixed cost. The profit function becomes

$$\text{(A20)} \qquad \Pi(\sigma) = \sum_{i=1}^{n} [p_i - c(q_i)]x_i - (1-\sigma)f(n).$$

Given n product varieties, the qualities and prices under a monopoly are

$$(q_i, p_i) \equiv \left(\frac{2i\bar{\theta}}{\alpha(2n+1)}, \frac{i(1+i+2n)\bar{\theta}^2}{\alpha(1+2n)^2}\right), \quad i = 1, ..., n.$$

Substituting them into equation (A20) yields

$$\Pi(\sigma) = \frac{n(1+n)\Gamma}{3(1+2n)^2} - (1-\sigma)f(n).$$

Therefore, with a fixed-cost subsidy, the optimal number of varieties $n(\sigma)$ under a monopoly is determined by

$$\text{(A21)} \qquad \frac{d\Pi(\sigma)}{dn} = \frac{\Gamma}{3(1+2n(\sigma))^3} - (1-\sigma)f'(n(\sigma)) = 0.$$

Equation (A15) implies that the first-best varieties are determined by

$$\frac{2\Gamma}{3(1+2n^F)^3} = f'(n^F).$$

Comparing this equation with equation (A21) leads to the conclusion that $\sigma_1 = 1/2$, $n(\sigma_1) = n^F$, and

$$(q_i(\sigma_1), p_i(\sigma_1)) \equiv \left(\frac{2i\bar{\theta}}{\alpha(2n^F+1)}, \frac{i(1+i+2n^F)\bar{\theta}^2}{\alpha(1+2n^F)^2}\right), \quad i = 1, ..., n^F.$$

Section A.4 indicates that the first-best qualities and prices are given by

$$(q_i^F, p_i^F) \equiv \left(\frac{2i\bar{\theta}}{\alpha(2n^F+1)}, \frac{2i^2\bar{\theta}^2}{\alpha(1+2n^F)^2}\right), \quad i = 1, ..., n^F.$$

Therefore, we have $q_i(\sigma_1) = q_i^F$ and

$$p_i(\sigma_1) - p_i^F = \frac{i(1+2n^F-i)\bar{\theta}^2}{\alpha(1+2n^F)^2} > 0, \quad i = 1, ..., n^F.$$

That is, a fixed-cost subsidy with subsidy rate $\sigma_1 = 1/2$ can induce both the first-best varieties and the first-best qualities, but the prices are much higher.

Equation (A16) implies that the second-best varieties are determined by

$$\Gamma/(2(1+2n^S)^3) = f'(n^S). \tag{A22}$$

Comparing equations (A21) and (A22) yields that $\sigma_2 = 1/3$, $n(\sigma_2) = n^S$, and $(q_i(\sigma_2), p_i(\sigma_2)) = (q_i^S, p_i^S)$, $i = 1, ..., n^S$. *Q.E.D.*

A.7 Proof of Proposition 4

(1) Proof of Proposition 4(1). Using the expression for x_i^* and combining equations (2), (9), (10), and (A5), we obtain firm j's profit and the second-best welfare function,

$$\Pi^{Oj} = \frac{\Phi(q_1, ..., q_n)}{(1+K)^2} - f(n), \quad j = 1, ..., K, \tag{A23}$$

$$W^{OS} = \frac{K(K+2)}{2(1+K)^2} \cdot \Phi(q_1, ..., q_n) - Kf(n), \tag{A24}$$

where

$$\Phi(q_1, ..., q_n) = \sum_{i=1}^{n} \left[\frac{1}{\mu_i} + \mu_i(c(q_i) - c(q_{i-1}))^2\right] - 2c(q_n).$$

The first-best welfare function is

$$W^{OF} = \sum_{i=1}^{n-1} \int_{(p_i-p_{i-1})\mu_i}^{(p_{i+1}-p_i)\mu_{i+1}} [\theta q_i - c(q_i)]d\theta + \int_{(p_n-p_{n-1})\mu_n}^{1} [\theta q_n - c(q_n)]d\theta - Kf(n).$$

Note that $p_i^F = c(q_i)$. The first-best welfare function can be simplified as

$$W^{OF} = \frac{1}{2} \cdot \Phi(q_1, ..., q_n) - Kf(n). \tag{A25}$$

Equations (A23)–(A25) imply that the optimal qualities $(q_1(n), ..., q_n(n))$ for these three equations are exactly the same. Substituting $(q_1(n), ..., q_n(n))$ into equations (A23)–(A25) leads to

$$\Pi^{Oj}(n) = \frac{1}{(1+K)^2} \cdot \Phi(q_1(n), ..., q_n(n), n) - f(n),$$

$$W^{OS} = \frac{K(K+2)}{2(1+K)^2} \cdot \Phi(q_1(n), ..., q_n(n), n) - Kf(n),$$
$$W^{OF}(n) = \frac{1}{2} \cdot \Phi(q_1(n), ..., q_n(n), n) - Kf(n).$$

Using the envelope theorem, we can show that product varieties under the monopoly, in the first-best and the second-best optimum, are respectively determined by

(A26) $$\frac{1}{(1+K)^2} \cdot \frac{\partial \Phi(n^{O*})}{\partial n} = f'(n^{O*}),$$

(A27) $$\frac{K+2}{2(1+K)^2} \cdot \frac{\partial \Phi(n^{OS})}{\partial n} = f'(n^{OS}),$$

(A28) $$\frac{1}{2K} \cdot \frac{\partial \Phi(n^{OF})}{\partial n} = f'(n^{OF}).$$

Combining equations (A27) and (A28) yields

$$\left(\frac{K+2}{2(1+K)^2} \cdot \frac{\partial \Phi(n^{OS})}{\partial n} - f'(n^{OS})\right) - \left(\frac{K+2}{2(1+K)^2} \cdot \frac{\partial \Phi(n^{OF})}{\partial n} - f'(n^{OF})\right)$$
$$= \frac{1}{2K(1+K)^2} \cdot \frac{\partial \Phi(n^{OF})}{\partial n}.$$

Using the mean-value theorem and equation (A28), we obtain

$$\left(\frac{K+2}{2(1+K)^2} \cdot \frac{d}{dn}\left(\frac{\partial \Phi(n_6)}{\partial n}\right) - f''(n_6)\right) \cdot (n^{OS} - n^{OF}) = \frac{f'(n^{OF})}{(1+K)^2} > 0, \quad n_6 \in (n^S, n^F).$$

The SOC of equation (A27) requires that

$$\frac{K+2}{2(1+K)^2} \cdot \frac{d}{dn}\left(\frac{\partial \Phi(n_6)}{\partial n}\right) - f''(n_6) < 0,$$

and hence $n^{OS} - n^{OF} < 0$. Similarly, we can show that $n^{O*} - n^{OS} < 0$.

(2) Proof of Proposition 4(2). Based on the cost function $c(q_i) = \alpha q_i^2/2$:

(a) We have

$$q_i^{OM} = \frac{2i}{\alpha(2n^{OM}+1)}, \quad M \in \{*, S, F\}, \; i = 1, ..., n^{OM};$$
$$q_{n^{OM}}^{OM} - q_1^{OM} = \frac{2(n^{OM}-1)}{\alpha(2n^{OM}+1)}.$$

Note that with

$$\frac{d\left(q_{n^{OM}}^{OM} - q_1^{OM}\right)}{dn^{OM}} = \frac{6}{\alpha(1+2n^{OM})^2} > 0 \quad \text{and} \quad n^{O*} < n^{OS} < n^{OF},$$

the proof of Proposition 4(2)(a) is completed, i.e., $q^{O*}_{n^{O*}} - q^{O*}_1 < q^{OS}_{n^{OS}} - q^{OS}_1 < q^{OF}_{n^{OF}} - q^{OF}_1$.

(b) We have

$$\Phi(n) = \frac{4n(n+1)}{3\alpha(2n+1)^2}.$$

Equations (A26)–(A28) can be simplified as

$$\text{(A29)} \qquad \frac{1}{(1+K)^2} \cdot \frac{4}{3(1+2n^{O*})^3\alpha} = f'(n^{O*}),$$

$$\text{(A30)} \qquad \frac{K+2}{2(1+K)^2} \cdot \frac{4}{3(1+2n^{OS})^3\alpha} = f'(n^{OS}),$$

$$\text{(A31)} \qquad \frac{1}{2K} \cdot \frac{4}{3(1+2n^{OF})^3\alpha} = f'(n^{OF}).$$

Under a variable-cost subsidy with subsidy rate $\tau > 0$, firms' marginal cost is $c(q_i) = \alpha(1-\tau)q_i^2/2$. The market provision of varieties, $n(\tau)$, is determined by

$$\text{(A32)} \qquad \frac{1}{(1-\tau)(1+K)^2} \cdot \frac{4}{3(1+2n(\tau))^3\alpha} = f'(n(\tau)).$$

Comparing equation (A32) with equations (A30) and (A31) leads to the following results:

(i) If

$$\frac{1}{(1-\tau)(1+K)^2} = \frac{1}{2K} \Leftrightarrow \tau_3 = \frac{1+K^2}{(1+K)^2},$$

then $n(\tau_3) = n^{OF}$.

(ii) If

$$\frac{1}{(1-\tau)(1+K)^2} = \frac{K+2}{2(1+K)^2} \Leftrightarrow \tau_4 = \frac{K}{K+2},$$

then $n(\tau_4) = n^{OS}$.

(c) Under a fixed-cost subsidy, firms' levels of quality and price do not change. Thus, given product varieties n and the cost function $c(q_i) = \alpha q_i^2/2$, we can use the symmetry to obtain firm i's output, quality, and price

$$x_i^{O*} = \frac{2}{(2n+1)(K+1)}, \quad q_i^{O*} = \frac{2i}{\alpha(2n+1)}, \quad \text{and} \quad p_i^{O*} = \frac{2i(1+Ki+2n)}{\alpha(1+K)(1+2n)^2}, \quad i = 1, ..., n^*.$$

The market provision of varieties, $n(\sigma)$, is determined by the following equation:

$$\text{(A33)} \qquad \frac{1}{(1+K)^2} \cdot \frac{4}{3(1+2n(\sigma))^3\alpha} = (1-\sigma)f'(n(\sigma)).$$

Comparing equation (A33) with equations (A30) and (A31) leads to the following results:

(i) If

$$\frac{1}{(1-\sigma)(1+K)^2} = \frac{1}{2K} \;\Leftrightarrow\; \sigma_3 = \frac{1+K^2}{(1+K)^2},$$

then $n(\sigma_3) = n^{OF}$.

(ii) If

$$\frac{1}{(1-\sigma)(1+K)^2} = \frac{K+2}{2(1+K)^2} \;\Leftrightarrow\; \sigma_4 = \frac{K}{K+2},$$

then $n(\sigma_4) = n^{OS}$.

Note that the quality effect does not create distortion, and a fixed-cost subsidy does not affect the levels of quality and price. If the number of varieties reaches the second-best optimum, the oligopoly solution is also the second-best solution.

Section A.4 implies that the first-best qualities and prices are given by

$$(q_i^F, p_i^F) = \left(\frac{2i}{\alpha(2n^F+1)}, \frac{2i^2}{\alpha(1+2n^F)^2}\right).$$

Therefore, we have $q_i^{O*}(\sigma_3) = q_i^F$ and

$$p_i^{O*}(\sigma_3) - p_i^F = \frac{2i(1-i+2n^F)}{\alpha(1+K)(1+2n^F)^2} > 0.$$

That is, a fixed-cost subsidy $\sigma_3 = [1+K^2]/[(1+K)^2]$ can induce both the first-best varieties and the first-best qualities, but the prices are much higher. *Q.E.D.*

A.8 Proof of Corollary 2

Section A.7 implies that

$$q_1^{OM} = \frac{2}{\alpha(2n^{OM}+1)} \quad \text{and} \quad q_{n^{OM}}^{OM} = \frac{2n^{OM}}{\alpha(2n^{OM}+1)}, \quad M \in \{*, S, F\}.$$

Note that

$$\frac{dq_1^{OM}}{dn^{OM}} = \frac{-4}{\alpha(2n^{OM}+1)^2} < 0 \qquad \text{and} \qquad \frac{dq_{n^{OM}}^{OM}}{dn^{OM}} = \frac{2}{\alpha(2n^{OM}+1)^2} > 0.$$

Observe, $n^{O*} < n^{OS} < n^{OF}$ implies that $q_1^{O*} > q_1^{OS} > q_1^{OF}$ and $q_{n^{O*}}^{O*} < q_{n^{OS}}^{OS} < q_{n^{OF}}^{OF}$. *Q.E.D.*